

A PRIORI ESTIMATES FOR A GENERALISED MONGE-AMPÈRE PDE ON SOME COMPACT KÄHLER MANIFOLDS

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ABSTRACT. We study a fully nonlinear PDE involving a linear combination of symmetric polynomials of the Kähler form on a Kähler manifold. A C^0 *a priori* estimate is proven in general and a gradient estimate is proven in certain cases. Independently, we also provide a method-of-continuity proof via a path of Kähler metrics to recover the existence of solutions in some of the known cases. Known results are then applied to an analytic problem arising from Chern-Weil theory and to a special Lagrangian-type equation arising from mirror symmetry.

1. INTRODUCTION

Consider the following fairly general equation on a compact Kähler manifold (X, ω) .

$$(1) \quad \omega_\phi^n = \sum_{k=1}^n \alpha_k \wedge \omega_\phi^{n-k},$$

where $\alpha_k \geq 0$ are closed smooth positive (k, k) -forms such that $\alpha_{k_0} > 0$ for at least one k_0 (in particular, k_0 can be equal to n), ϕ is a smooth function such that $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, and ω satisfies the so-called “cone condition” $n\omega^{n-1} - \sum_{k=1}^n (n-k)\alpha_k \wedge \omega^{n-k-1} > 0$ and the consistency condition $\int \omega^n = \sum_{k=1}^n \int \alpha_k \wedge \omega^{n-k}$. Our notion of positivity of (p, p) -forms is explained in section 2.

Notice that if $\alpha_k = \delta_{kn}\eta$ then equation 1 boils down to the Calabi-Yau theorem [23]. In its full generality, equation 1 and its cousins arise in the representation problem of Chern-Weil theory [17], canonical metrics in Kähler geometry [21, 20, 11, 12, 10, 8, 6], symplectic geometry [18] and mirror symmetry in string theory [13, 15]. So far, in most places where it has been studied a flow technique (like the J-flow) was used to study it. In Wei Sun’s paper [20] the method of continuity was used but it used non-Kähler metrics to prove openness. In this paper we aim to prove *a priori* estimates and solve 1 in some cases.

Our first result is an *a priori* C^0 estimate on ϕ under general assumptions (proposition 2.6). We remark that a paper of Székelyhidi [21] deals with an equation that overlaps with the generalised Monge-Ampère equation in some cases (like the J-flow case). In [21] the ABP estimate is used to prove a C^0 -estimate but we use Yau’s Moser iteration argument. In the course of setting up the method of continuity we indicate a proof of a theorem (theorem 1.4 in [20] when ψ is a constant which is also

a theorem of Fang-Lai-Ma [11]) in remark 2.4 using a method of continuity that uses only Kähler metrics. This may potentially be of independent interest. Then we proceed to prove a gradient estimate for 1 in a special case.

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold. Assume that χ is another Kähler form such that χ has non-negative bisectional curvature. Assume that $\alpha = f\chi^{n-1} + (\sqrt{-1})^{n-1}(-1)^{n(n-1)/2} \sum_{a=1}^N f_a \Phi_a \wedge \bar{\Phi}_a$ is a closed smooth $(n-1, n-1)$ -form (N is an arbitrary natural number) which satisfies $-C\alpha \leq \nabla_X \alpha \leq C\alpha$ for all real χ -unit vectors X , where $f \geq 0, f_a \geq 0$ are smooth functions, Φ_a are smooth $(n-1, 0)$ -forms, $C > 0$ is a constant, and ∇ is the canonical connection induced by χ . Let $\eta > 0$ be an (n, n) -form. Assume that $\int \omega^n = \int \alpha \wedge \omega + \int \eta$ and that $n\omega^{n-1} - \alpha > 0$. For a smooth function ϕ , denote $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. Consider the equation.*

$$(2) \quad \omega_\phi^n = \alpha \wedge \omega_\phi + \eta.$$

The following hold.

- (1) Gradient estimate A smooth solution ϕ of equation 2 satisfies $\|\phi\|_{C^1} \leq C$ where C depends only on the coefficients.
- (2) Partial Laplacian estimate and existence If α is parallel with respect to χ then $\frac{\alpha \wedge \omega_\phi}{\chi^n} \leq C$. In addition, if $\alpha > 0$, i.e., if $f > 0$, then a unique smooth solution exists satisfying $\omega_\phi > 0$ and $n\omega_\phi^{n-1} - \alpha > 0$.

Remark 1.2. The existence part of theorem 1.1 actually follows from a far more general theorem of Wei Sun [20]. However, the *a priori* estimates are new in the case of α being degenerate. This is perhaps the main point of the theorem. The assumptions of theorem 1.1 might look restrictive but actually even as stated the technique used to prove them might potentially be useful in [18] where similar assumptions are in force (essentially [18] deals with domains in \mathbb{R}^n but the difficulty is that the equation there is degenerate elliptic).

In situations involving equations like 1 dealing with the J-flow or in the special case of 1 mentioned in remark 2.4 one can actually avoid the gradient estimate by proving the laplacian estimate directly. In fact, thanks to the work of Tosatti and Weinkove [22], just proving the estimate $\Delta\phi \leq Ce^{A(\phi - \inf \phi)}$ is enough to guarantee a C^2 bound on ϕ . Usually the technique behind proving such estimates is to use the maximum principle on an appropriately chosen function. For instance, one choice [20] is $\psi = e^w(\Delta_\chi \phi + \text{tr}_\chi \omega)$ where $w = -A\phi + f(\phi)$ is chosen judiciously. This method was pioneered by Aubin [1] and Yau [23]. The major difficulty here is that in general, equation 1 is not a symmetric polynomial in the hessian. This problem is exacerbated if we allow α_k to be degenerate. Therefore it is not clear that some inequalities in the spirit of [20, 6, 11] work in this setting.

Independently, we apply the main result in [20] to prove two theorems. The first one deals with Chern-Weil theory.

Theorem 1.3. *Let (V, h_0) be a hermitian rank- k holomorphic vector bundle over a compact Kähler manifold (X, ω) . Denote the curvature of the Chern connection of h_0 by F_0 and define $\Theta_0 = \frac{\sqrt{-1}F_0}{2\pi}$. Let η be an (n, n) -form on X representing*

the top Chern character class, i.e., $[\eta] = [\text{tr}(\Theta_0)^n]$. Define the forms α_i inductively according to

$$\begin{aligned} \alpha_1 &= n \left(\omega - \frac{1}{k} \text{tr}(\Theta_0) \right) \\ \alpha_p &= - \binom{n}{p} \frac{1}{k} \text{tr}(\Theta_0)^p + \binom{n}{p} \omega^p - \sum_{i=1}^{p-1} \binom{n-i}{p-i} \alpha_i \omega^{p-i} \quad \forall 2 \leq p \leq n-1 \\ (3) \quad \alpha_n &= \frac{\eta}{k} - \text{tr}(\Theta_0)^n + \omega^n - \sum_{i=1}^{n-1} \alpha_i \omega^{n-i}. \end{aligned}$$

Assume that there exists a hermitian metric χ , constants $c_i \geq 0$, and a smooth function ψ such that $\alpha_i = c_i \chi^i \psi$ and $\sum c_i > 0$. Also assume that ω satisfies $n\omega^{n-1} - \sum_{i=1}^{n-1} (n-i) \alpha_i \omega^{n-i-1} > 0$. Then there exists a smooth metric $h = h_0 e^{-2\pi\phi}$, unique upto constant multiples such that the top Chern character form of h is η , i.e., $\eta = \text{tr}(\Theta_h)^n$.

Some examples of the applicability of a very restricted version of theorem 1.3 are given in [17].

The second one deals with a special-Lagrangian type equation motivated from mirror symmetry.

Theorem 1.4. *Let L be a holomorphic line bundle over a compact Kähler manifold (X, ω) . Let $\hat{\theta}$ be defined by the equation $\text{Im}([\omega] + 2\pi\sqrt{-1}c_1(L))^3 = \tan(\hat{\theta})\text{Re}([\omega] + 2\pi\sqrt{-1}c_1(L))^3$. Assume that $\tan(\hat{\theta}) > 0$. Also assume that there exists a metric h_0 on L whose curvature F_0 is such that the $(1,1)$ -form $\Omega = \sqrt{-1}F_0 - \omega \tan(\hat{\theta})$ satisfies*

- (1) $\Omega > 0$, and
- (2) $\Omega^2 - \omega^2 \sec^2 \hat{\theta} > 0$.

Then there exists a smooth metric $h = h_0 e^{-\phi}$, unique upto constant multiples satisfying

$$(4) \quad \text{Im}(\omega - F_\phi)^3 = \tan(\hat{\theta}) \text{Re}(\omega - F_\phi^3).$$

We remark that since the theorem 1.4 does not require non-negative bisectional curvature, it is in some cases more general than the result in [13]. In particular, it may be applied to the Calabi-Yau 3-folds that are of interest to physicists. We give an example in section 5.

Here is a more detailed outline of the paper. In section 2 we set up the method of continuity, prove uniqueness and a uniform estimate, indicate a proof of the theorem in [20], and also prove that upper bounds on ω_ϕ lead to uniform ellipticity. Owing to the non-symmetric nature of the equation, this is actually somewhat non-trivial. In sections 3 and 4 we prove further *a priori* estimates in the special case of the equation in theorem 1.1. In section 5 we prove theorems 1.3 and 1.4.

2. SETUP OF THE METHOD OF CONTINUITY AND THE UNIFORM ESTIMATE

Before proceeding further, we define a notion of positivity of (p, p) -forms.

Definition 2.1. Let (X, χ) be a hermitian manifold. A smooth (p, p) -form α_p is (strictly) positive if $\alpha_p = f\chi^p + (\sqrt{-1})^p(-1)^{p(p-1)/2} \sum_{k=1}^N f_k \Phi_k \wedge \bar{\Phi}_k$ where f is a (strictly) positive smooth function, $f_k \geq 0$ are positive¹ smooth functions, and Φ_k are smooth $(p, 0)$ -forms. Moreover, we write $\alpha \geq 0$ if α is positive and $\alpha > 0$ if it is strictly so.

Remark 2.2. Perhaps a more natural definition would be to require that α_p define a hermitian non-negative bilinear form on $\Lambda^p T^{(1,0)}X$. However, one can easily see that this is equivalent to our definition. In particular, the wedge product of strictly positive forms is strictly positive.

In order to solve 1 we employ the method of continuity. In whatever follows we assume that on (X, ω) for at least one value of k_0 , $\alpha_{k_0} > \delta \omega^{k_0}$ for a positive constant δ . Consider the following family of equations parametrised by $t \in [0, 1]$.

$$(5) \quad \omega_{\phi_t}^n = t \sum_{k=1}^{n-1} \alpha_k \wedge \omega_{\phi_t}^{n-k} + \alpha_n b_t c^{1-t},$$

where $\omega_{\phi_t} = \omega + \sqrt{-1} \partial \bar{\partial} \phi_t$ is a Kähler form, α_k are d -closed positive (k, k) -forms and α_n is a strictly positive (n, n) -form, $c = \frac{\int \omega^n}{\int \alpha_n}$, and b_t is a normalising constant chosen so that the integrals are equal on both sides, i.e., $b_t = c^{t-1} \frac{\int \left(\omega^n - t \sum_{k=1}^{n-1} \alpha_k \wedge \omega^{n-k} \right)}{\int \alpha_n} = c^{t-1} \frac{\int ((1-t)\omega^n + t\alpha_n)}{\int \alpha_n}$. Thus $b_t c^{1-t} \geq 1$.

Let \mathcal{T} be the set of $t \in [0, 1]$ where equation 5 has a unique smooth solution ϕ_t such that $\int \phi_t \omega^n = 0$, $\omega_{\phi_t} > 0$, and $n\omega_{\phi_t}^{n-1} - t \sum (n-k) \alpha_k \wedge \omega_{\phi_t}^{n-k-1} > 0$. \mathcal{T} is non-empty because at $t = 0$ the equation is the usual Monge-Ampère equation which has a solution thanks to [23]. As usual, we need to prove that \mathcal{T} is both, open and closed.

Openness : Let \mathcal{C} be the set of $C^{2,\beta}$ zero-average functions ϕ such that $\omega_\phi > 0$ where the background metric used to define the Banach spaces and the average is ω . We proceed to define a smooth map T from \mathcal{B} (where \mathcal{B} is an open subset of $\mathcal{C} \times [0, 1]$ such that $n\omega_\phi^{n-1} - t \sum (n-k) \alpha_k \wedge \omega_\phi^{n-k-1} > 0$) to $C^{0,\beta}$ top forms

$$\gamma > 0 \text{ such that } \int \gamma = 1. \text{ It is given by } T(\phi, t) = \frac{\omega_\phi^n - t \sum_{k=1}^{n-1} \alpha_k \omega_\phi^{n-k}}{\int \omega^n - t \sum_{k=1}^{n-1} \int \alpha_k \omega^{n-k}}. \text{ The}$$

derivative DT at the point (ϕ_a, a) evaluated on the vector $(u, 0)$ is computed to be $DT_{\phi_a, a}(u, 0) = \frac{(n\omega_{\phi_a}^{n-1} - a \sum (n-k) \alpha_k \wedge \omega_{\phi_a}^{n-k-1}) \wedge dd^c u}{\int \omega^n - a \sum \int \alpha_k \omega^{n-k}}$. It is easily seen to be a self-adjoint elliptic operator. By the Fredholm alternative, we can solve the PDE if the

¹Unless specified otherwise, we use positive in the french sense to mean non-negative

right hand side is orthogonal to its kernel. Its kernel (by the maximum principle) consists of constants. Thus by the implicit function theorem on Banach manifolds, on the level set $T^{-1}\left(\frac{\alpha_n}{f^{\alpha_n}}\right)$ we can locally solve for ϕ as a smooth function of t .

Closedness : If $t_j \rightarrow t$, we need to prove that a subsequence $\phi_j \rightarrow \phi$ in $C^{2,\gamma}$, $\omega_\phi > 0$, and $n\omega_\phi^{n-1} - t \sum (n-k)\alpha_k \wedge \omega_\phi^{n-k-1} > 0$. By the usual bootstrap argument this implies that ϕ is smooth. The Arzela-Ascoli theorem shows that it is enough to prove *a priori* $C^{2,\gamma}$ estimates in order to show convergence of $\phi_j \rightarrow \phi$. The following argument shows that the limiting ϕ satisfies the other conditions.

Lemma 2.3. *If $\|\phi_j\|_{C^2} \leq C$, then $\omega_{\phi_j} \geq R\omega > 0$ and $n\omega_{\phi_j}^{n-1} - t_j \sum (n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \geq R\omega^{n-1} > 0$ where the positive constant R depends on C .*

Proof. Recall that by assumption $\alpha_{k_0} \geq \delta\omega^{k_0}$ for some k_0 and some constant $\delta > 0$. Equation 5 implies that

$$1 \geq \delta \frac{\omega_{\phi_j}^{k_0} \wedge \omega_{\phi_j}^{n-k_0}}{\omega_{\phi_j}^n} \geq \delta \frac{k_0!(n-k_0)!}{n!} \sum \frac{1}{\lambda_{i_1} \dots \lambda_{i_{k_0}}}.$$

where λ_i are the eigenvalues of ω_{ϕ_j} with respect to ω . Hence we see that $\omega_{\phi_j} \geq R\omega$ for some $R > 0$ depending on the upper bound on ω_{ϕ_j} . Indeed, if the smallest eigenvalue λ_1 (with respect to ω) of ω_{ϕ_j} becomes arbitrarily small then the right hand side of the equation above becomes arbitrarily large because ω_{ϕ_j} is bounded above by assumption.

Likewise, at an arbitrary point p , let v be a unit $(1,0)$ -form with respect to ω . Choose coordinates so that ω_{ϕ_j} is diagonal with eigenvalues 1 and $v = c \frac{\partial}{\partial z^1}$ at p . Note that c is bounded below and above because ω_{ϕ_j} is. Then

$$\begin{aligned} \left(n\omega_{\phi_j}^{n-1} - t_j \sum_k (n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right) \wedge v \wedge \bar{v} &= c^2 \left(n\omega_{\phi_j}^{n-1} - t_j \sum_k (n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right)_{2\bar{2}3\bar{3}\dots n\bar{n}} \\ &= c^2 n! - c^2 t_j \sum_k \sum_{|I|=n-k-1, 1 \notin I, I^0} (n-k)! (\alpha_k)_{I^0}, \end{aligned}$$

where if the multi-index $I = (i_1, \dots, i_{n-k-1})$ then I^0 is the multi-index consisting of k other numbers in $1, 2, \dots, n$. Equation 5 implies that

$$\begin{aligned} n! &= t_j \sum_k \sum_{|V|=n-k} (n-k)! (\alpha_k)_{V^0} \\ \Rightarrow n! - t_j \sum_k \sum_{|I|=n-k-1, 1 \notin I, I^0} (n-k)! (\alpha_k)_{I^0} &= t_j \sum_k \sum_{|W|=n-k, 1 \notin W} (n-k)! (\alpha_k)_{W^0} \\ &\geq t_j \delta (n-k_0)! \sum_{|W|=n-k_0, 1 \notin W} (\omega^{k_0})_{W^0} \geq \tilde{R}. \end{aligned}$$

The last equation implies that $\left(n\omega_{\phi_j}^{n-1} - t_j \sum_k (n-k)\alpha_k \wedge \omega_{\phi_j}^{n-k-1} \right) \wedge v \wedge \bar{v} \geq R$ for some $R > 0$ depending on the upper bound on ω_{ϕ_j} . \square

Remark 2.4. At this juncture, if in equation 1 we substitute $\alpha_k = \psi_\epsilon c_k \chi^k \forall 1 \leq k \leq n-1$, $\alpha_n = (c_n + \epsilon)\chi^n$ (where χ is a Kähler metric), such that $c_k \geq 0 \forall 1 \leq$

$k \leq n$, $\epsilon > 0$ and $\psi_\epsilon = \frac{\int \omega^n - \int (\epsilon + c_n)\chi^n}{\int \omega^n - \int c_n\chi^n}$ are constants such that $\sum_{k=1}^n c_k > 0$ and

$\int \omega^n = \sum_{k=1}^n \int c_k \chi^k \wedge \omega^{n-k}$, then the $C^{2,\gamma}$ *a priori* estimates in [20, 6] guarantee that

the resulting equation has a smooth solution ϕ_ϵ . We note that the *a priori* estimates up to the second order in [20, 6] do not depend on ϵ . Actually, using the Evans-Krylov theorem and the fact that (by lemma 2.3) the equation is uniformly elliptic we have $C^{2,\gamma}$ *a priori* estimates independent of ϵ . Therefore, upto a subsequence $\phi_\epsilon \rightarrow \phi$ in $C^{2,\beta}$ as $\epsilon \rightarrow 0$. Hence we recover the main theorem in [20] in the Kähler case via a continuity path that passes only through Kähler metrics and more importantly, openness is easy to prove (as opposed to [20]).

As mentioned earlier, lemma 2.3 shows that all we have to do in order to solve equation 1 is to prove *a priori* $C^{2,\gamma}$ estimates on ϕ . We prove a general C^0 estimate on ϕ here.

The uniform estimate : Before proceeding further, we prove a lemma about concavity of certain potentially non-symmetric functions of the Kähler form.

Lemma 2.5. *The function $\omega \rightarrow \frac{\alpha_k \wedge \omega^{n-k}}{\omega^n}$ is a convex function of Kähler forms if $\alpha_k \geq 0$.*

Proof. Recall that $\alpha_k = f\chi^k + \sqrt{-1}^{n-1}(-1)^{n(n-1)/2} \sum_{i=1}^N f_i \Phi_i \wedge \bar{\Phi}_i$ where $f, f_i \geq 0$.

Choosing normal coordinates for χ we see that $\omega \rightarrow f \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}$ is convex by standard theory. Furthermore, let ω_1 and ω_2 be two Kähler forms. At the point under consideration choose coordinates so that ω_1 is Euclidean and ω_2 is diagonal with eigenvalues λ_j . Therefore for some positive constant C we have,

$$(6) \quad \frac{\Phi_i \wedge \bar{\Phi}_i \wedge (t\omega_1 + (1-t)\omega_2)^{n-k}}{(t\omega_1 + (1-t)\omega_2)^n} = C \sum_{|I|=n-k} (\Phi_i \wedge \bar{\Phi}_i)_{I^0} \frac{(t + (1-t)\lambda)_I}{(t + (1-t)\lambda_1) \dots (t + (1-t)\lambda_n)}.$$

It is now easy to deduce the desired result from expression 6 and the fact that $\frac{1}{\det(A)}$ is convex as a function of positive-definite matrices A . \square

Proposition 2.6. *A smooth solution of equation 1 satisfies $\|\phi\|_{C^0(X)} \leq C$ where C depends only on ω , bounds on the coefficients of the equation, and the positive lower bound on α_{k_0} .*

Proof. We follow Yau's by-now-classical [23] technique adapted from [16]. In what follows, unless otherwise specified, all controlled constants are denoted by C . Without loss of generality we may change the normalisation of ϕ so that $\sup \phi = -1$. Let $\phi = -\phi_-$. We will find an upper bound on ϕ_- using Moser iteration (as usual). Let $\Theta = \omega^n - \sum_{k=1}^n \alpha_k \wedge \omega^{n-k}$. Subtracting Θ on both sides

from $\omega_\phi^n - \sum_{k=1}^n \alpha_k \wedge \omega_\phi^{n-k} = 0$, multiplying by ϕ_-^p and integrating we see that

$$\begin{aligned} & \int_X \phi_-^p \int_0^1 \frac{d}{dt} \left(\omega_{t\phi}^n - \sum_{k=1}^n \alpha_k \wedge \omega_{t\phi}^{n-k} \right) dt \leq C \|\phi_-\|_{L^p}^p \\ \Rightarrow & - \int_0^1 \int_X \phi_-^p \left(n \omega_{t\phi}^{n-1} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right) \sqrt{-1} \partial \bar{\partial} \phi_- dt \leq C \|\phi_-\|_{L^p}^p \\ \Rightarrow & \int_0^1 \int_X \sqrt{-1} \partial \phi_-^{\frac{p+1}{2}} \wedge \bar{\partial} \phi_-^{\frac{p+1}{2}} \wedge \left(n \omega_{t\phi}^{n-1} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right) dt \\ & \leq \frac{C(p+1)^2}{p} \|\phi_-\|_{L^p}^p \leq C(p+1) \|\phi_-\|_{L^p}^p. \end{aligned}$$

At a point q we choose coordinates normal coordinates z^i for ω so that $\partial \phi_-$ is proportional to $\frac{\partial}{\partial \bar{z}^1}$. This means that at q

$$\begin{aligned} & \partial \phi_-^{\frac{p+1}{2}} \wedge \bar{\partial} \phi_-^{\frac{p+1}{2}} \wedge \left(n \omega_{t\phi}^{n-1} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right) \\ & = \partial \phi_-^{\frac{p+1}{2}} \bar{\partial} \phi_-^{\frac{p+1}{2}} \left(n \omega_{t\phi}^{n-1} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right)_{2\bar{2} \dots n\bar{n}} \\ & = \partial \phi_-^{\frac{p+1}{2}} \bar{\partial} \phi_-^{\frac{p+1}{2}} \left(n \omega_{t\phi}^{n-1} \right)_{2\bar{2} \dots n\bar{n}} \left(1 - \frac{\left(\sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right)_{2\bar{2} \dots n\bar{n}}}{\left(n \omega_{t\phi}^{n-1} \right)_{2\bar{2} \dots n\bar{n}}} \right). \end{aligned}$$

Now we restrict ourselves to the subspace spanned by $\partial_2, \dots, \partial_n$. To emphasize this we denote the restriction of any form β by $\tilde{\beta}$. Now we proceed as in [20]. Note that $\omega_{t\phi} = t\omega_\phi + (1-t)\omega$ and that $\det(A)^{1/n}$ is concave as a function of positive-definite $n \times n$ matrices A . Therefore $\tilde{\omega}_{t\phi}^{n-1} \geq t^{n-1} \tilde{\omega}_\phi^{n-1} + (1-t)^{n-1} \tilde{\omega}^{n-1}$. Likewise, lemma 2.5 shows that

$$\begin{aligned} 1 - \frac{\left(\sum_{k=1}^n (n-k) \alpha_k \wedge \omega_{t\phi}^{n-k-1} \right)_{2\bar{2} \dots n\bar{n}}}{\left(n \omega_{t\phi}^{n-1} \right)_{2\bar{2} \dots n\bar{n}}} & \geq t \left(1 - \frac{\sum_{k=1}^n (n-k) \tilde{\alpha}_k \wedge \tilde{\omega}_\phi^{n-k-1}}{n \tilde{\omega}_\phi^{n-1}} \right) \\ & + (1-t) \left(1 - \frac{\sum_{k=1}^n (n-k) \tilde{\alpha}_k \wedge \tilde{\omega}^{n-k-1}}{n \tilde{\omega}^{n-1}} \right) \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{n+1} \int_X \sqrt{-1} \partial \phi_-^{\frac{p+1}{2}} \wedge \bar{\partial} \phi_-^{\frac{p+1}{2}} \wedge \left(n \omega^{n-1} - \sum_{k=1}^n (n-k) \alpha_k \wedge \omega^{n-k-1} \right) \leq C(p+1) \|\phi_-\|_{L^p}^p \\ & \Rightarrow \|\nabla(\phi_-^{(p+1)/2})\|_{L^2}^2 \leq C(p+1) \|\phi_-\|_{L^p}^p. \end{aligned}$$

From this point onwards, the proof is standard. (See [16] for instance.) \square

Uniqueness : If ϕ_1 and ϕ_2 are two smooth solutions of equation 1 such that $\int \phi_1 \omega^n = \int \phi_2 \omega^n$, $\omega_{\phi_i} > 0$ and $n\omega_{\phi}^{n-1} - \sum_{k=1}^{n-1} (n-k)\alpha_k \wedge \omega_{\phi_i}^{n-k} > 0$, then upon subtraction we get

$$(7) \quad \int_0^1 \frac{d}{dt} \left(\omega_{t\phi_1+(1-t)\phi_2}^n - \sum_{k=1}^n \alpha_k \wedge \omega_{t\phi_1+(1-t)\phi_2}^{n-k} \right) dt = 0$$

$$\Rightarrow \left(\int_0^1 \left(n\omega_{t\phi_1+(1-t)\phi_2}^{n-1} - \sum_{k=1}^{n-1} (n-k)\alpha_k \wedge \omega_{t\phi_1+(1-t)\phi_2}^{n-k-1} \right) dt \right) \wedge \sqrt{-1}\partial\bar{\partial}(\phi_1 - \phi_2) = 0$$

The proof of proposition 2.6 shows that equation 7 is elliptic. Thus the maximum principle implies that $\phi_1 = \phi_2$.

3. THE GRADIENT ESTIMATE

From now onwards we restrict ourselves to solving a special case of equation 1 on the Kähler manifold (X, ω) where ω satisfies the cone condition. Firstly, let $\frac{1}{b}\omega \leq \chi \leq b\omega$ be an arbitrary Kähler metric on X having nonnegative bisectional curvature. As mentioned in the introduction, we aim at solving

$$(8) \quad \omega_{\phi}^n = \alpha \wedge \omega_{\phi} + \eta,$$

where $\alpha = f\chi^{n-1} + \sum_{a=1}^N f_a \Phi_a \wedge \bar{\Phi}_a$ and $\eta > 0$. In addition we assume that $-C\alpha \leq \nabla_X \alpha \leq C\alpha$ where X is a real χ -unit vector. Also, from now onwards we write $\eta = h\chi^n$ where $h > 0$ is a smooth function, $\alpha \wedge \beta$ locally as $\chi^n A^{k\bar{l}} \beta_{k\bar{l}}$ for a non-negative matrix A , and ω locally as $\omega_{i\bar{j}} dz^i d\bar{z}^j$ where ω is used (by abuse of notation) to denote both, the Kähler potential as well as the metric itself.

In order to prove a gradient bound on ϕ we use Blocki's technique [2]. Denote by ∇ the Levi-Civita connection associated to χ . Let $\psi = \ln(|\nabla\phi|^2) - \gamma(\phi)$ where $\gamma(t) = \frac{1}{2} \ln(2t+1)$ is chosen so that $\gamma' > E > 0$ and $-(\gamma'' + (\gamma')^2) > Q > 0$ for two positive constants E and Q . At the maximum point p of ψ , $\nabla\psi = 0$ and $\psi_{k\bar{l}}$ is negative semi-definite. Without loss of generality we may assume that $|\nabla\phi|(p) \geq N$ for any N . Choosing normal coordinates for χ at p so that ω_{ϕ} is diagonal with eigenvalues λ_i we obtain,

$$(9) \quad 0 = \psi_k(p) = \frac{\sum_i \phi_{ik} \phi_{\bar{i}} + \phi_i \phi_{\bar{i}k}}{|\nabla\phi|^2} - \gamma' \phi_k$$

$$\psi_{k\bar{l}}(p) = -(\gamma'' + (\gamma')^2) \phi_k \phi_{\bar{l}} - \gamma' \phi_{k\bar{l}} + \frac{\sum_i \phi_{ik\bar{l}} \phi_{\bar{i}} + \phi_{ik} \phi_{\bar{i}\bar{l}} + \phi_{ik} \phi_{i\bar{l}} + \phi_i \phi_{\bar{i}k\bar{l}}}{|\nabla\phi|^2} + \frac{\chi_{,k\bar{l}}^{i\bar{j}} \phi_i \phi_{\bar{j}}}{|\nabla\phi|^2}.$$

Rewriting equation 8 as $1 = \frac{\alpha\omega_\phi}{\omega_\phi^n} + \frac{\eta}{\omega_\phi^n}$, differentiating once, and multiplying by $\phi_{\bar{i}}$, at the point p we obtain the following (after using the assumption on $\nabla\alpha$).

$$(10) \quad 0 \geq -L^{k\bar{l}}\phi_{\bar{i}}(\omega_{k\bar{l}i} + \phi_{k\bar{l}i}) - C|\nabla\phi|$$

where $L^{k\bar{l}} = -\frac{A^{k\bar{l}}}{\lambda_1 \dots \lambda_n} + \frac{\delta_{k\bar{l}}}{\lambda_k}$. Multiplying equation 9 by $L^{k\bar{l}}$ and using equation 10 we obtain,

$$(11) \quad 0 \geq QL^{k\bar{l}}\phi_k\phi_{\bar{l}} - \gamma' L^{k\bar{l}}(\omega_\phi)_{k\bar{l}} + \gamma' L^{k\bar{l}}\omega_{k\bar{l}} + L^{k\bar{l}}\frac{\chi_{k\bar{l}}^{i\bar{j}}\phi_i\phi_{\bar{j}}}{|\nabla\phi|^2} - \frac{\sum_i L^{k\bar{l}}(\omega_{k\bar{l}i}\phi_{\bar{i}} + \omega_{k\bar{l}i}\phi_i)}{|\nabla\phi|^2} - \frac{C}{|\nabla\phi|}.$$

Noting that $L^{k\bar{l}}(\omega_\phi)_{k\bar{l}} = -\frac{\alpha\omega_\phi}{\omega_\phi^n} + n$ (which is larger than $n-1$ and less than n), and the assumption on the bisectional curvature $\chi_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0 \forall \alpha, \beta$ we get

$$0 \geq QL^{k\bar{l}}\phi_k\phi_{\bar{l}} + \gamma' L^{k\bar{l}}\omega_{k\bar{l}} - \gamma' \left(n - \frac{\alpha\omega_\phi}{\omega_\phi^n} \right) - 2\frac{|L^{k\bar{l}}\nabla\omega_{k\bar{l}}|}{|\nabla\phi|} - \frac{C}{|\nabla\phi|}.$$

Now we multiply on both sides by $\frac{\omega_\phi^n}{n!}$ and define $\tilde{L}^{k\bar{l}} = \frac{\omega_\phi^n}{n!}L^{k\bar{l}} = A^{k\bar{l}} - \delta^{k\bar{l}}\frac{\lambda_1 \dots \lambda_n}{\lambda_k}$. We get

$$(12) \quad \begin{aligned} 0 &\geq Q\tilde{L}^{k\bar{l}}\phi_k\phi_{\bar{l}} + \left(\gamma' - \frac{C}{|\nabla\phi|} \right) \tilde{L}^{k\bar{l}}\omega_{k\bar{l}} - \gamma' \left(n(h + A^{k\bar{l}}(\omega_\phi)_{k\bar{l}}) - A^{k\bar{l}}(\omega_\phi)_{k\bar{l}} \right) - \frac{C\omega_\phi^n}{|\nabla\phi|} \\ &\Rightarrow C \geq Q\tilde{L}^{k\bar{l}}\phi_k\phi_{\bar{l}} + \left(\gamma' - \frac{C}{|\nabla\phi|} \right) \left(-A^{k\bar{l}}\omega_{k\bar{l}} + \frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n}(h + A^{k\bar{l}}(\omega_\phi)_{k\bar{l}}) \right) \\ &\quad - (n-1)\gamma' A^{k\bar{l}}(\omega_\phi)_{k\bar{l}} - \frac{C(h + A^{k\bar{l}}(\omega_\phi)_{k\bar{l}})}{|\nabla\phi|}. \end{aligned}$$

Note that inequality 12 implies that at p , the expression $\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n}$ is bounded above.

At p if we can prove that $\tilde{L}^{k\bar{l}} \geq T\chi^{k\bar{l}} > 0$ then we will have a gradient estimate on ϕ . Actually, if we manage to prove that $\Delta_\chi\phi$ is bounded above, then by lemma 2.3 we are done. If we just prove that $\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} > n-1+\epsilon$ for some uniform positive constant ϵ then inequality 12 implies that ω_ϕ^n is bounded above and hence by the lower bound on ω_ϕ so is $\Delta_\chi\phi$ above. Indeed, the following lemma coupled with this observation completes the proof of the gradient estimate.

Lemma 3.1. *If at a point q , $\omega_\phi \geq R\omega > 0$, $\Delta_\chi\phi \rightarrow \infty$ then $\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} > n-1+\epsilon$ for some uniform positive constant ϵ .*

Proof. Without loss of generality we assume that $\lambda_1 \geq \lambda_2 \dots \lambda_n$. Equation 8 implies that at least λ_n is bounded above at p . Notice that the cone condition implies

$$(13) \quad (n\omega^{n-1})_{2\bar{2} \dots n\bar{n}} > n!A^{1\bar{1}}.$$

By the classical Hadamard inequality for matrices (see [7] for instance) $\omega_{2\bar{2}} \dots \omega_{n\bar{n}} \geq \frac{(\omega^{n-1})_{2\bar{2} \dots n\bar{n}}}{(n-1)!}$. Thus

$$(14) \quad \omega_{2\bar{2}} \dots \omega_{n\bar{n}} > A^{1\bar{1}}.$$

Solving for λ_1 from equation 8 we see that $\lambda_1 = \frac{f + \sum_{k \neq 1} A^{k\bar{k}}}{\lambda_2 \dots \lambda_n - A^{1\bar{1}}}$. This coupled with the lower bound on the λ_i and the assumption that $\lambda_1 \rightarrow \infty$ implies that $\lambda_2 \dots \lambda_n \rightarrow A^{1\bar{1}}$. Therefore,

$$\begin{aligned} \frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} &= \sum_{k=1}^n \frac{\omega_{k\bar{k}}}{\lambda_k} \rightarrow \sum_{k=2}^n \frac{\omega_{k\bar{k}}}{\lambda_k} \\ &\geq (n-1) \left(\prod_{k=2}^n \frac{\omega_{k\bar{k}}}{\lambda_k} \right)^{1/(n-1)} \rightarrow (n-1) \left(\frac{\prod_{k=2}^n \omega_{k\bar{k}}}{A^{1\bar{1}}} \right)^{1/(n-1)}, \end{aligned}$$

where we used the AM-GM inequality. Using inequality 14 we are done. \square

4. HIGHER ORDER ESTIMATES

In this section we prove the partial Laplacian estimate. In addition to the assumptions in section 3 we assume that α is parallel with respect to χ .

Remark 4.1. It is but natural to wonder whether there are any forms α that satisfy the desired requirements other than multiples of χ^{n-1} . If X is a complex torus and χ the flat metric, then $\alpha = dz^1 \wedge d\bar{z}^1 \dots dz^{n-1} \wedge d\bar{z}^{n-1}$ furnishes a non-trivial degenerate example. In general, one can take a locally hermitian symmetric space or a product of any 2 manifolds with χ being the product metric to produce lots of examples using $(1,1)$ -forms. (Note that in our case we also need χ to have nonnegative bisectional curvature.) In fact, it is known that on manifolds other than local products or locally hermitian symmetric spaces the only such forms are indeed multiples of χ^{n-1} . According to Bryant [4], this result follows from the classification of Riemannian holonomy groups.

Partial Laplacian bound : We now prove an upper bound on $\frac{\alpha\omega_\phi}{\chi^n}$. As in [17] we use the function $\Psi = \frac{\alpha\omega_\phi}{\chi^n} - \mu\phi$ where μ is a constant that will be chosen later. If we prove that Ψ is bounded above then we are done. As before, at the maximum point p of Ψ , $\Psi_k = 0$ and $\Psi_{k\bar{l}}$ is negative semi-definite. We choose normal coordinates for χ at p and make sure that ω_ϕ is diagonal at p with eigenvalues λ_i . Differentiation of Ψ yields the following.

$$\begin{aligned} 0 = \nabla_k \Psi &= \Psi_k = \frac{\alpha\omega_{\phi,k}}{\chi^n} - \mu\phi_k \\ \Psi_{k\bar{l}} &= \nabla_{\bar{l}} \nabla_k \Psi = \frac{\alpha\nabla_{\bar{l}} \nabla_k \omega_\phi}{\chi^n} - \mu\nabla_{\bar{l}} \nabla_k \phi \\ &\geq \frac{\alpha\omega_{\phi,k\bar{l}}}{\chi^n} - \mu(\omega_\phi)_{k\bar{l}} + \mu\omega_{k\bar{l}} \end{aligned} \tag{15}$$

Differentiating $1 = \frac{\alpha\omega_\phi}{\omega_\phi^n} + \frac{\eta}{\omega_\phi^n}$ twice, multiplying by $A^{k\bar{l}}$ and summing over $k = l$ we obtain

$$(16) \quad 0 = \frac{\alpha\omega_{\phi,k}}{\omega_\phi^n} - \frac{n\omega_\phi^{n-1}\omega_{\phi,k}}{\omega_\phi^n} + \frac{\eta_{,k}}{\omega_\phi^n}$$

$$(17) \quad 0 \geq \frac{A^{k\bar{l}}\alpha\nabla_{\bar{l}}\nabla_k\omega_\phi}{\omega_\phi^n} - \frac{A^{k\bar{l}}n\omega_\phi^{n-1}\nabla_{\bar{l}}\nabla_k\omega_\phi}{\omega_\phi^n} - A^{k\bar{l}}\frac{\eta_{,k}}{\omega_\phi^n}\frac{n\omega_\phi^{n-1}\omega_{\phi,\bar{l}}}{\omega_\phi^n} - A^{k\bar{l}}\frac{\eta_{,\bar{l}}}{\omega_\phi^n}\frac{n\omega_\phi^{n-1}\omega_{\phi,k}}{\omega_\phi^n} + \frac{\nabla_{\bar{l}}\nabla_k\eta}{\omega_\phi^n}$$

$$\geq \frac{A^{k\bar{l}}\alpha\omega_{\phi,k\bar{l}}}{\omega_\phi^n} - \frac{A^{k\bar{l}}n\omega_\phi^{n-1}\omega_{\phi,k\bar{l}}}{\omega_\phi^n} - C\left(\frac{\alpha\omega_\phi}{\omega_\phi^n} + n\right) - A^{k\bar{l}}\frac{\eta_{,k}}{\omega_\phi^n}\frac{n\omega_\phi^{n-1}\omega_{\phi,\bar{l}}}{\omega_\phi^n} - A^{k\bar{l}}\frac{\eta_{,\bar{l}}}{\omega_\phi^n}\frac{n\omega_\phi^{n-1}\omega_{\phi,k}}{\omega_\phi^n} - \frac{C\eta}{\omega_\phi^n},$$

where we used lemma 2.5. At this juncture we use equations 15 and 16 to get

$$(18) \quad C \geq \frac{A^{k\bar{l}}\alpha\omega_{\phi,k\bar{l}}}{\omega_\phi^n} - \frac{A^{k\bar{l}}n\omega_\phi^{n-1}\omega_{\phi,k\bar{l}}}{\omega_\phi^n} - \mu C\left(\frac{\chi^n}{\omega_\phi^n}\right)^2.$$

We multiply equation 15 by $L^{k\bar{l}}$ and sum to obtain (after substituting in 18)

$$(19) \quad \begin{aligned} C &\geq -\mu C\left(\frac{\chi^n}{\omega_\phi^n}\right)^2 - L^{k\bar{l}}\mu(\omega_\phi)_{k\bar{l}} + L^{k\bar{l}}\mu\omega_{k\bar{l}} \\ &\Rightarrow \frac{C}{\mu} \geq -C\left(\frac{\chi^n}{\omega_\phi^n}\right)^2 - \left(n - \frac{\alpha\omega_\phi}{\omega_\phi^n}\right) + \left(\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} - \frac{\alpha\omega}{\omega_\phi^n}\right) \\ &= -C\left(\frac{\chi^n}{\omega_\phi^n}\right)^2 - \left(n\frac{\eta}{\omega_\phi^n} + (n-1)\frac{\alpha\omega_\phi}{\omega_\phi^n}\right) + \left(\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} - \frac{\alpha\omega}{\omega_\phi^n}\right) \end{aligned}$$

Since $\eta > 0$ we know that ω_ϕ^n is bounded from below. Moreover, $0 \leq \frac{\alpha\omega_\phi}{\omega_\phi^n} \leq 1$.

Therefore $\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} < C$. This implies a lower bound, $\omega_\phi > R\omega > 0$. Since we are assuming that $\Delta_\chi\phi \rightarrow \infty$,

$$\frac{C}{\mu} \geq \frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} + (1-n)\frac{\alpha\omega_\phi}{\omega_\phi^n}$$

Using lemma 3.1 we see that $\frac{n\omega_\phi^{n-1}\omega}{\omega_\phi^n} > n-1+\epsilon$ for some uniform positive constant ϵ . Therefore $\frac{C}{\mu} \geq \epsilon$. Choosing μ to be large enough we arrive at a contradiction. This proves the partial Laplacian estimate.

In the case when $\alpha > 0$ the partial Laplacian estimate implies an estimate on $\Delta_\chi\phi$.

$C^{2,\gamma}$ estimates : The previously established partial Laplacian bound when $\alpha > 0$ implies by lemma 2.3 that the equation is uniformly elliptic. In fact, it also implies that $1 = \frac{\alpha\omega_\phi}{\omega_\phi^n} + \frac{\eta}{\omega_\phi^n}$ is uniformly elliptic. Lemma 2.5 implies that the equation is also convex. Thus the (complex version of) the Evans-Krylov theory [19] is applicable and furnishes a $C^{2,\gamma}$ estimate. This completes the proof of theorem 1.1.

5. APPLICATIONS

5.1. Representation of the top Chern character.

Given a (k, k) form η representing the k th Chern character class $[tr((\Theta)^k)]$ of a vector bundle on a compact complex manifold (where $\frac{\sqrt{-1}F}{2\pi} = \Theta$ and F is the curvature of a connection), it is natural to ask whether there is a metric h on the vector bundle whose induced Chern connection realises

$$(20) \quad \text{tr}(\Theta^k) = \eta.$$

As phrased this question seems almost intractable. It is not even obvious as to whether there is *any* connection satisfying this requirement, leave aside a Chern connection. Work along these lines was done by Datta in [9] using the h-principle. Therefore, it is more reasonable to ask whether equality can be realised for the top Chern character form. To restrict ourselves further we ask whether any given metric h_0 may be conformally deformed to $h = h_0 e^{-\phi}$ satisfying the desired requirement. In the case of a line bundle L (where the only choice we have is conformal deformations) equation 20 boils down to the PDE

$$\text{tr} \left(\Theta_0 + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^n = \eta$$

When $\Theta_0 > 0$ and $\eta > 0$ this is the usual Monge-Ampère equation solved by Yau [23]. In general, one gets a complicated fully nonlinear PDE which reduces to equation 1 in some cases. It is clear from the case of a line bundle that for the general case of a vector bundle, unfortunately quite a few potentially unnatural positivity requirements will have to be made on the curvature Θ_0 and the form η . Note that the local problem was dealt with in [14]. In [17] an existence result was proven on complex 3-tori. Using the result in [20] we prove theorem 1.3.

Proof of theorem 1.3: According to theorem 1.1 in [20] the equation

$$(21) \quad \omega_\phi^n = e^b \psi \sum_1^n c_i \chi^i \omega_\phi^{n-i}$$

on a compact Kähler manifold (X, ω) has a unique smooth solution ϕ, b satisfying $\omega_\phi > 0$ and $n\omega_\phi^{n-1} - \psi \sum_{i=1}^{n-1} (n-i)c_i \chi^i \omega_\phi^{n-i-1} > 0$ if there exists a smooth function v such that

$$(22) \quad \omega_v^n \leq \psi \sum_{i=i}^n c_i \chi^i \omega_v^{n-i}.$$

Actually, if $\int_X \omega^n = \int_X \psi \sum_1^n c_i \chi^i \omega^{n-i}$ and $\psi \chi^i$ is closed, then one can choose $v = 0$ and $\tilde{\psi} = e^{-\tilde{b}} \psi$ where b is small enough for condition 22 to hold. This shows that under such circumstances, equation 21 has a unique solution with $b = 0$. Indeed,

expanding equations 3 and 21 (with $b = 0$) we get

$$\begin{aligned}
 & \sum_{r=0}^n \binom{n}{r} \frac{1}{k} \operatorname{tr}(\Theta_0^r) (\sqrt{-1} \partial \bar{\partial} \phi)^{n-r} - \frac{\eta}{k} = 0 \\
 & \sum_{r=0}^n \omega^r (\sqrt{-1} \partial \bar{\partial} \phi)^{n-r} - \psi \sum_{k=1}^n \sum_{r=0}^{n-k} c_k \chi^k \binom{n-k}{r} \omega^r (\sqrt{-1} \partial \bar{\partial} \phi)^{n-k-r} = 0
 \end{aligned}
 \tag{23}$$

Comparing the two equations we get the desired result. \square

5.2. A special Lagrangian type equation.

According to superstring theory the spacetime of the universe is constrained to be a product of a compact Calabi-Yau three-fold and a four dimensional Lorentzian manifold. A “duality” relates the geometry of this Calabi-Yau manifold with another “mirror” Calabi-Yau manifold. From a differential geometry standpoint this maybe thought of (roughly) as a relationship between the existence of “nice” metrics on a line bundle on one Calabi-Yau manifold and special Lagrangian submanifolds of the other Calabi-Yau manifold. Using the Fourier-Mukai transform, Leung-Yau-Zaslow showed [15] that this mirror symmetry implies that equation 4 ought to be satisfied in some cases. In [13], Jacob and Yau showed that given an ample line bundle L over a compact Kähler manifold with non-negative orthogonal bisectional curvature, L^k admits a solution to equation 4. However, the assumption of non-negative orthogonal bisectional curvature is not desirable if one wants to apply such a result to general Calabi-Yau manifolds. Here we attempt to partially address this issue by restricting our attention to 3-folds.

Proof of theorem 1.4 : Equation 4 can be written using $\Theta = \sqrt{-1}F$ as

$$-\Theta^3 + 3\omega^2\Theta = \tan(\hat{\theta}) (\omega^3 - 3\Theta^2\omega).$$

Grouping terms together we see that it is equivalent to

$$(24) \quad \Omega_\phi^3 - 3\omega^2\Omega_\phi \sec^2(\hat{\theta}) - 2\omega^3 \tan(\hat{\theta}) \sec^2(\hat{\theta}) = 0,$$

where $\Omega_\phi = \Omega + \sqrt{-1} \partial \bar{\partial} \phi = \Theta_0 - \omega \tan(\hat{\theta}) + \sqrt{-1} \partial \bar{\partial} \phi$. Comparing this equation to the theorem in [20] we see that if $\Omega > 0$, $\tan(\hat{\theta}) > 0$, and the cone condition $\Omega^2 - \omega^2 \sec^2(\hat{\theta}) > 0$ is satisfied, then the equation has a unique smooth solution upto a constant multiple. \square

The conditions imposed on Ω in theorem 1.4 are reminiscent of the “stability” condition in [13]. Here is a concrete example of a Calabi-Yau manifold where the theorem is applicable :

Let X be $\mathbb{C}\mathbb{A} \times K$ where K is a projective K3 surface with Picard group generated by an ample line bundle L (for example K can be a non-singular degree 4 surface in \mathbb{P}^3). Let ω be the product of the flat metric on the torus and the unique Calabi-Yau metric in the Kähler class $[L] + \epsilon[\gamma]$ where $[\gamma]$ is any cohomology class and ϵ is chosen to be small enough (as to how small can be determined easily) for the

example to work. Endow L with a metric h_0 with positive curvature F_0 . Choose k to be large enough so that for (L^k, h_0^k) , $\tan(\hat{\theta}_k) > 0$. Indeed,

$$\tan(\hat{\theta}_k) = \frac{\int (k^3 \Theta_0^3 - 3k \Theta_0 (\Theta_0 + \epsilon \gamma)^2)}{\int (3k^2 \Theta_0^2 (\Theta_0 + \epsilon \gamma) - (\Theta_0 + \epsilon \gamma)^3)}.$$

So if $k \geq 2$ and ϵ small enough, then $\tan(\hat{\theta}) > 0$. Note that as $k \rightarrow \infty$, $\tan(\hat{\theta}_k)$ grows linearly in k .

Notice that if $\Theta_0 > 2\omega \tan(\hat{\theta})$ then the cone condition is definitely satisfied for large enough k . Indeed,

$$\begin{aligned} \Omega_0^2 - \omega^2 \sec^2(\hat{\theta}) &= \Theta_0^2 + \omega^2 \tan^2(\hat{\theta}) - 2\Theta_0 \omega \tan(\hat{\theta}) - \omega^2 \sec^2(\hat{\theta}) \\ &= \Theta_0^2 - 2\Theta_0 \omega \tan(\hat{\theta}) - \omega^2 = \Theta_0(\Theta_0 - 2\omega \tan(\hat{\theta})) - \omega^2 \end{aligned}$$

Hence, our requirement boils down to making sure that $k\Theta_0 - 2\omega \tan(\hat{\theta}_k) > 0$ for large k .

$$(25) \quad k\Theta_0 - 2\omega \tan(\hat{\theta}_k) = k\Theta_0 - 2(\Theta_0 + \epsilon \gamma) \frac{\int (k^3 \Theta_0^3 - 3k \Theta_0 (\Theta_0 + \epsilon \gamma)^2)}{\int (3k^2 \Theta_0^2 (\Theta_0 + \epsilon \gamma) - (\Theta_0 + \epsilon \gamma)^3)}$$

For large enough k we just have to look at

$$(26) \quad k\Theta_0 - \frac{2}{3}k(\Theta_0 + \epsilon \gamma) \frac{\int \Theta_0^3}{\int \Theta_0^2 (\Theta_0 + \epsilon \gamma)}$$

which is obviously positive if ϵ is small enough.

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REFERENCES

- [1] T. Aubin, Métriques Riemanniennes et courbure. *Journal of Differential Geometry*, 4.4 (1970): 383-424.
- [2] Z. Blocki, The Calabi-Yau theorem. *Lecture Notes in Mathematics 2038*, 2005 (also on <http://gamma.im.uj.edu.pl/~blocki/publ/>).
- [3] Z. Blocki, A gradient estimate in the Calabi-Yau theorem. *Mathematische Annalen*, 344.2 (2009): 317-327.
- [4] R. Bryant, <http://mathoverflow.net/questions/204467/>
- [5] E. Calabi, The space of Kähler metrics. *Proceedings of the International Congress of Mathematicians, Amsterdam*, 1954.
- [6] T. Collins and G. Székelyhidi, *Convergence of the J-flow on toric manifolds*, arXiv : 1412.4809 (preprint).
- [7] T. Cover and J. Thomas, Determinant inequalities via information theory. *SIAM journal of matrix analysis and applications*, 9(3) (1988) : 384-392.
- [8] X.X. Chen, On the lower bound of the Mabuchi energy and its application. *International Mathematical Research Notices*, 2000.12 (2000): 607-623.
- [9] M. Datta, Universal property of Chern character forms of the canonical connection. *Geometric and Functional Analysis*, 14.6 (2004): 1219-1237.

- [10] S. Dinew and S. Kolodziej. A priori estimates for complex Hessian equations. *Analysis and PDE*, 7.1 (2014): 227-244.
- [11] F. Hao, M. Lai, X. Ma, On a class of fully nonlinear flows in Kähler geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2011.653 (2011): 189-220.
- [12] Z. Hou, X.N. Ma, D. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold. *Mathematical Research Letters*, 17.2 (2010): 547-562.
- [13] A. Jacob and S.T. Yau, A special Lagrangian type equation for holomorphic line bundles. arXiv : 1411.7457 (preprint)
- [14] N.V. Krylov, On a general notion of fully nonlinear second-order elliptic equations. *Transactions of the American Mathematical Society*, 347.3 (1995): 857-895.
- [15] C. Leung, S-T Yau, and E. Zaslow, From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform. *Winter school on mirror symmetry, vector bundles, and Lagrangian submanifolds (Cambridge, MA, 1999)* : 209-225.
- [16] V. Pingali, A generalised Monge-Ampère equation. arXiv: 1205.1266 (To appear in the Journal of Partial differential equations)
- [17] V. Pingali, A fully nonlinear generalized Monge-Ampère PDE on torus. *Electronic Journal of Differential Equations*, Vol. 2014 (211) (2014) : 1-8.
- [18] Y. Rubinstein and J. Solomon, The degenerate special Lagrangian equation. arXiv:1506.08077.
- [19] Y.T. Siu, Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics. *Birkhäuser*, 1987.
- [20] W. Sun, Generalized complex Monge-Ampère type equations on closed Hermitian manifolds. arXiv: 1412.8192.
- [21] G. Székelyhidi, Fully nonlinear elliptic equations on compact Hermitian manifolds. arXiv:1501.02762.
- [22] V. Tosatti and B. Weinkove. Estimates for the complex Monge-Ampere equation on Hermitian and balanced manifolds. *Asian Journal of Mathematics* 14.1 (2010): 19-40.
- [23] S.T. Yau, On the Ricci curvature of a compact kähler manifold and the complex monge-ampère equation, I. *Communications in Pure and Applied Mathematics*, 31.3 (1978): 339-411.

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